# Solutions

**Exam 1**
**Chapter 5.3-5.5 and 8**

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Answer the following questions. *You must show your work to receive full credit.* Be sure to make reasonable simplifications. Indicate your final answer with a $\boxed{\text{box}}$.

**1.** (10 points) Evaluate the indefinite integral $\int e^x \cos x dx$.

$u = e^x \quad dv = \cos x dx$

$du = e^x dx \quad v = \sin x$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$u' = e^x \quad dv' = \sin x dx$

$du' = e^x dx \quad v' = -\cos x$

$$= e^x \sin x - \left(-e^x \cos x + \int e^x \cos x dx\right)$$

$$= e^x(\sin x + \cos x) - \int e^x \cos x dx.$$

Thus

$$2\int e^x \cos x dx = e^x(\sin x + \cos x) + C$$

and

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + C.$$

2. (10 points) Evaluate the definite integral $\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^3 x dx$.

Solution 1: $\cos x$ is even and thus $\cos^3 x$ is even.

$\sin x$ is odd and thus $\sin^3 x$ is odd.

Thus $\sin^3 x \cos^3 x$ is odd and $\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^3 x \, dx = 0$.

Solution 2:

$$\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^3 x \, dx = \int_{-\pi/2}^{\pi/2} \sin x (1-\cos^2 x) \cos^3 x \, dx \qquad \begin{array}{l} u = \cos x \\ du = -\sin x dx \end{array}$$

$$= -\int_{-\cancel{\pi/2}}^{0} (1-u^2) u^3 \, du$$

$$= -\left(\frac{u^4}{4} - \frac{u^6}{6}\right)\Big|_{0}^{0}$$

$$= -\left(\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right)\Big|_{-\pi/2}^{\pi/2} = 0.$$

Solution 3: Same as Solution 2, except you write $\cos^3 x = \cos x(1-\sin^2 x)$ and let $u = \sin x$.

**3.** (10 points) Find $\frac{dy}{dx}$, where

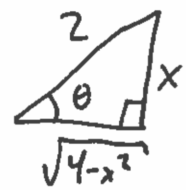
$$y = \int_1^x t^3 e^{t^2}\, dt.$$

$$\frac{dy}{dx} = x^3 e^{x^2}.$$

**4.** (10 points) Find the area between the curve given by $y = \frac{dx}{(4-x^2)^{3/2}}$ and the $x$-axis between $x = 0$ and $x = 1$.

$$\text{Area} = \int_0^1 \frac{dx}{(4-x^2)^{3/2}}$$

Let $\cancel{x} = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$




$$= \int_D^D \frac{2\cos\theta\, d\theta}{\cancel{\sqrt{}}(4(1-\sin^2\theta))^{3/2}}$$

$$= \frac{1}{4}\int_D^D \frac{\cos\theta\, d\theta}{\cos^3\theta} = \frac{1}{4}\int_D^D \sec^2\theta\, d\theta = \frac{1}{4}\tan\theta \Big|_D^D$$

$$= \frac{1}{4}\frac{x}{\sqrt{4-x^2}}\Big|_0^1 = \frac{1}{4}\left(\frac{1}{\sqrt{3}} - 0\right) = \frac{\sqrt{3}}{12}.$$

**5.** (10 points) Evaluate the indefinite integral $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$.

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)\,(x+1)^2}.$$

Thus $x^2 = A(x^2+2x+1) + B(x^2-1) + C(x-1)$

$= (A+B)x^2 + (2A+C)x + (A-B-C)$

and $1 = A+B$

$0 = 2A+C$

$0 = A-B-C.$

Thus $A = -\frac{C}{2}$. $0 = -\frac{C}{2} - B - C \Rightarrow B = -\frac{3C}{2}$.

So $1 = -\frac{C}{2} - \frac{3C}{2} = -2C$ and $C = -\frac{1}{2}$, $B = \frac{3}{4}$, and $A = \frac{1}{4}$.

Now $\int \frac{x^2 dx}{(x-1)(x^2+2x+1)} = \int \frac{1}{4(x-1)}\,dx + \int \frac{3}{4(x+1)}\,dx - \int \frac{1}{2(x+1)^2}\,dx$

$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C.$

**6.** (10 points) Determine if the improper integral $\int_{-8}^{1} \frac{dx}{x^{1/3}}$ converges. If it does, evaluate the integral.

$\frac{1}{x^{1/3}}$ is not defined at zero. So

$$\int_{-8}^{1} \frac{dx}{x^{1/3}} = \lim_{a \to 0^-} \int_{-8}^{a} \frac{dx}{x^{1/3}} + \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{x^{1/3}}$$

$$= \lim_{a \to 0^-} \frac{3}{2} x^{2/3} \Big|_{-8}^{a} \lim_{b \to 0^+} \frac{3}{2} x^{2/3} \Big|_{b}^{1}$$

$$= \lim_{a \to 0^-} \frac{3}{2}\left(a^{2/3} - (-8)^{2/3}\right) + \lim_{b \to 0^+} \frac{3}{2}\left(1 - b^{2/3}\right)$$

$$= \frac{3}{2}(0-4) + \frac{3}{2}(1-0) = -\frac{9}{2}.$$

The integral converges to $-\frac{9}{2}$.